# ON A GENERALIZATION OF THE METHOD OF ELASTIC SOLUTIONS

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D. L. BYKOV and V. A. SHACHNEV
(Moscow)
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A method of construction of a process of successive approximations based on the use of an effective shear modulus is proposed for solution of problems of the theory of small elastic-plastic strains. This method permits solution of problems for materials in which the region of linearity between stress and strain is either absent or very small. It is shown herein that the choice of effective modulus is restricted by several conditions which, when violated, can lead to divergent processes. In the case of an incompressible material, the convergence of the successive approximations to the generalized solution of the first and the second boundary value problems is proved under the restrictions mentioned.

In an example it is shown that the limitations required for convergence of the approximations in the general case can be relaxed in the solution of specific problems.

As is well known [1] problems of the theory of small elastic-plastic deformations reduce to finding the stresses $\sigma_{kj}$, strains $e_{kj}$, and displacements $u_j$, which satisfy the equations of equilibrium, the stress-strain and stress-displacement laws, and also the boundary conditions

$$\sigma_{jk,\,k} + X_j = 0 \qquad (j, k = 1, 2, 3) \tag{1}$$

$$\sigma_{jk} - \sigma\delta_{jk} = \frac{2\sigma_i}{3e_i}(e_{jk} - e\delta_{jk}), \qquad e_{jk} = {}^1\!/_2 (u_{j,\,k} + u_{k,\,j}) \tag{2}$$

$$\sigma = 3Ke, \qquad \sigma_i = 3Ge_i\,[1 - \omega(e_i)] \tag{3}$$

$$u_j\,|_\Gamma = u_{j0}, \qquad \sigma_{jk} l_k\,|_\Gamma = \sigma_{j0} \tag{4}$$

Here the symbol $(\ )_{,j}$ denotes differentiation with respect to the Cartesian coordinate $x_j$ of a point in the region $\Omega$, which is occupied by the body; summation is to be carried out on repeated subscripts; $\sigma_i$ and $e_i$ are the intensities of shearing stress and strain; $\sigma$ and $e$ are the mean stress and strain; $\delta_{jk}$ is the Kronecker delta; $K$ is the bulk modulus; $G$ is the shear modulus; $X_j$ are the body forces; $\sigma_{j0}$ and $u_{j0}$ are the surface tractions and displacements; and $l_k$ are the direction cosines of the normal to the surface $\Gamma$ which is the boundary of the finite region $\Omega$.

The first and second of the boundary conditions (4) determine, respectively, the first and second boundary value problems of the theory of plasticity for the system of equations (1) – (3).

The function $\omega(e_i)$ introduced by Il'iushin in the equation relating to shear stress and shear strain intensities [1] permits application of the method of elastic solutions owing to the fact that for strain hardening materials

$$0 \leqslant \omega(e_i) \leqslant \omega(e_i) + \frac{\omega(e_i) - \omega(e_i')}{e_i - e_i'}\,e_i' = \frac{d\omega(e_i^\circ)\,e_i^\circ}{de_i^\circ} \leqslant \eta < 1 \tag{5}$$

The smallness of the parameter $\omega$, which ensures the convergence of the process of successive approximations [2 and 3], allows an especially effective solution of the problem when the deviations from elastic strains in the body are not very large. As the function $\omega$ increases the convergence of the successive approximations still obtains, but the rate of convergence decreases. It is, therefore, expedient to rewrite the relation between $\sigma_i$ and $e_i$ when solving problems involving considerable departures from elastic strains in such a way that the convergence of the successive approximations might be better than in the usual method of elastic solutions. This can be guaranteed if instead of the usual shear modulus $G$, some "effective" shear modulus $G^*$ is used.

It is obvious from physical considerations that after development of plastic strains, the "mean" shear modulus in the body will be smaller than the purely elastic modulus $G$; therefore, $G^*$ should also be taken smaller than $G$.

We introduce the function $\tau(e_i)$ as follows:

$$\tau(e_i) = 1 - [1 - \omega(e_i)]\, G/G^* \qquad (G^* > 0) \tag{6}$$

so that for $\sigma_i$ we obtain, in accordance with (2) and (6),

$$\sigma_i = 3G^* e_i [1 - \tau(e_i)] \tag{7}$$

The following inequalities then hold for the function $\tau(e_i)$:

$$1 - \frac{G}{G^*} \leqslant \frac{\tau(e_{i2})\, e_{i2} \pm \tau(e_{i1})\, e_{i1}}{e_{i2} \pm e_{i1}} \leqslant 1 - \frac{a}{3G^*}, \qquad \frac{\tau(e_{i2}) - \tau(e_{i1})}{e_{i2} - e_{i1}} \geqslant 0 \tag{8}$$

They follow from the condition

$$3G \geqslant (\sigma_{i2} \pm \sigma_{i1}) / (e_{i2} \pm e_{i1}) \geqslant a > 0, \qquad \sigma_i(0) = 0$$

and from the condition of convexity of the curve $\sigma_i(e_i)$, which is true for a wide class of experimental $\sigma_i$ vs. $e_i$ relationships.

Taking account of (7), we can represent the equations of equilibrium in terms of displacements in the form

$$(K + {}^1\!/_3 G^*)\, u_{k,\,kj} + G^* u_{j,\,kk} = - X_j + G^* [\tau (u_{j,\,k} + u_{k,\,j})]_{,\,k} - {}^2\!/_3 G^* (\tau u_{k,\,k})_{,\,j} \qquad (k, j = 1, 2, 3) \tag{9}$$

where the double scripts after a comma signify a second derivative with respect to the corresponding coordinates.

We define a sequence of approximate solutions by the following recurrence relations in a manner similar to the method of elastic solutions:

$$(K + {}^1\!/_3 G^*)\, u_{k,\,kj}^{(n+1)} + G^* u_{j,\,kk}^{(n+1)} = - X_j + G^* [\tau_n (u_{j,\,k}^{(n)} + u_{k,\,j}^{(n)})]_{,\,k} - {}^2\!/_3 G^* (\tau_n u_{k,k}^{(n)})_{,j} \tag{10}$$

$$\tau_n = \tau(e_i^{(n)})$$

The functions $u_k^{(n+1)}$ satisfy the first of the conditions (4) in the case of the first boundary value problem, and the condition

$$[(K - {}^2\!/_3 G^*)\, u_{m,\,m}^{(n+1)} \delta_{jk} + G^* (u_{j,\,k}^{(n+1)} + u_{k,\,j}^{(n+1)})]\, l_k |_\Gamma = \sigma_{j0} + G^* \tau_n [u_{j,\,k}^{(n)} + u_{k,\,j}^{(n)} - {}^2\!/_3 u_{m,\,m}^{(n)} \delta_{jk}]\, l_k |_\Gamma \tag{11}$$

in the case of the second boundary value problem.

Let us consider the conditions under which the sequence (10) with the appropriate boundary conditions converges to a solution of Eq. (9). In what follows we shall assume that the boundary condition (4) in the case of the first boundary value problem can be made homogeneous

$$u_j |_\Gamma = 0 \tag{12}$$

as a result of changing the body forces.

For arbitrary differentiable vector functions $\mathbf{u} = u_j \mathbf{e}_j$ and $\mathbf{v} = v_j \mathbf{e}_j$ ($\mathbf{e}_j$ is the unit vector in the direction of the $x_j$-axis), we define the scalar product and norm at a point by the expressions

$$(\mathbf{u}, \mathbf{v}) = {}^3\!/_4 (u_{j,\,k} + u_{k,\,j})(v_{j,\,k} + v_{k,\,j}) - \theta_u \theta_v, \qquad \theta_u = u_{m,\,m}, \qquad \theta_v = v_{m,\,m} \tag{13}$$

$$\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})} = {}^3\!/_2 \sqrt{2} e_i \tag{14}$$

It is easy to verify that the axioms for the scalar product [4] are thereby satisfied, except for one which will not be used in what follows: it does not follow from $\| \mathbf{u} \| = 0$ that $\mathbf{u} = 0$.

Considering the system (9) as a vector equation, we form its scalar product with a continuously differentiable function $\mathbf{v}$ and integrate over the region $\Omega$. For the first boundary value problem we then obtain Eq.

$$K \int \theta_u \theta_v \, d\Omega + \frac{2}{3} G^* \int (\mathbf{u}, \mathbf{v}) \, d\Omega = \frac{2}{3} G^* \int \tau (\| \mathbf{u} \|) (\mathbf{u}, \mathbf{v}) \, d\Omega + \int X_j v_j \, d\Omega \tag{15}$$

For the second boundary value problem we have an equation of the form

$$K \int \theta_u \theta_v \, d\Omega + \frac{2}{3} G^* \int (\mathbf{u}, \mathbf{v}) \, d\Omega = \frac{2}{3} G^* \int \tau (\| \mathbf{u} \|) (\mathbf{u}, \mathbf{v}) \, d\Omega + \int X_j v_j \, d\Omega + \int \sigma_{j0} v_j \, d\Gamma \tag{16}$$

It is obvious that a solution of Eq. (9) for the appropriate boundary conditions satisfies Eq. (15) or (16). However, Eq. (9) does not always have a solution for the boundary conditions indicated. This is related to the fact that in the classical formulation the problem is improperty posed. At the same time, for these problems Eq. (15) or (16) admits a unique solution, which it is natural to consider as the solution of the problem which has been formulated. We call such a solution a generalized solution of Eq. (9).

We shall examine the solution of Eq. (15) or (16) in a special function space where the scalar product and norm are defined for differentiable functions in accordance with the expressions

$$(\mathbf{u}, \mathbf{v})_\Omega = \int (\mathbf{u}, \mathbf{v}) \, d\Omega, \qquad \| \mathbf{u} \|_\Omega = \sqrt{(\mathbf{u}, \mathbf{u})_\Omega} \tag{17}$$

In this same space we introduce in addition the norm $\| \mathbf{u} \|_{1\Omega}$, generated by the scalar product

$$(\mathbf{u}, \mathbf{v})_{1\Omega} = \int (\mathbf{u}, \mathbf{v})_1 \, d\Omega, \qquad (\mathbf{u}, \mathbf{v})_1 = (\mathbf{u}, \mathbf{v}) + \frac{3K}{2G^*} \theta_u \theta_v \tag{18}$$

It is easy to show the equivalence of the norms $\| \mathbf{u} \|_\Omega$ and $\| \mathbf{u} \|_{1\Omega}$, for instance, for the set of vector functions satisfying the condition (12), if the Stokes representation is used for the displacements $u_j$.

We shall determine the solution of the problem in the space H which is obtained by the closure relative to the norm (17) of the set of twice continuously differentiable vector functions, satisfying the condition (12) in the case of the first boundary value problem. There is no need to satisfy the boundary conditions in the case of the second boundary value problem, since they are contained in (16) itself. It can be seen from Korn's inequality [4] that functions the generalized derivatives of which are square summable belong to H. In connection with this, we examine the condition of boundedness in H of the linear functionals

$$\int X_j v_j d\Omega, \quad \int \sigma_{j0} v_j d\Gamma$$

which occur in (15) and (16).

In accordance with the imbedding theorems [5], they hold if

$$X_j \in L_p(\Omega), \quad p \geqslant 6/5, \qquad \sigma_{j0} \in L_q(\Gamma), \quad q > 4/3$$

We show that under these conditions a solution of Eq. (15) exists in H (this result is given in [2]). In what follows all the transformations will be carried out only for the first boundary value problem, since for the second boundary value problem all the proofs are analogous in form.

Let us form the scalar product of (10) with a vector function $\mathbf{v}$ which is twice continuously differentiable and is equal to zero on the boundary of the region $\Omega$. Integrating over $\Omega$, we set up the recurrence relation

$$K\int \theta_u^{(n+1)}\theta_v\,d\Omega+\frac{2}{3}G^*\int(\mathbf{u}^{(n+1)},\mathbf{v})\,d\Omega=\frac{2}{3}G^*\int\tau(\|\mathbf{u}^{(n)}\|)(\mathbf{u}^{(n)},\mathbf{v})\,d\Omega+\int X_j v_j\,d\Omega \quad (19)$$

The successive solutions of this relation in the space H are the generalized solutions of the sequence (10), and they exist if the right-hand side of (19) is a functional which is bounded in H on the set of vector functions $\mathbf{v}$ dense in H. This last is true, taking account of the condition on $X_j$ for a suitable choice of the initial element. For instance, it is sufficent that the vector function $\mathbf{u}^{(0)}$ be piecewise differentiable. For convenience we shall assume that, as before, the function $\tau(e_i)$ is differentiable. (The existence of a solution to the relation (19) is easily proven by using the known results [4] on the boundedness of the operator on the left-hand side of (19) from below by a constant in the space H).

By the Schwartz inequality, the triangle inequality, and the definition (14), we obtain the following from (19):

$$|(\mathbf{u}^{(n+1)}-\mathbf{u}^{(n)},\mathbf{v})_{1\Omega}|=\left|\int\tau(\|\mathbf{u}^{(n)}\|)(\mathbf{u}^{(n)},\mathbf{v})-\tau(\|\mathbf{u}^{(n-1)}\|)(\mathbf{u}^{(n-1)},\mathbf{v})\,d\Omega\right|=$$

$$=\left|\int\tau(\|\mathbf{u}^{(n)}\|)(\mathbf{u}^{(n)}-\mathbf{u}^{(n-1)},\mathbf{v})+[\tau(\|\mathbf{u}^{(n)}\|)-\tau(\|\mathbf{u}^{(n-1)}\|)](\mathbf{u}^{(n-1)},\mathbf{v})\,d\Omega\right|\leqslant$$

$$\leqslant\int\left[|\tau(e_i)|+\frac{\tau(e_i^{(n)})-\tau(e_i^{(n-1)})}{e_i^{(n)}-e_i^{(n-1)}}e_i^{(n-1)}\right]\|\mathbf{u}^{(n)}-\mathbf{u}^{(n-1)}\|\|\mathbf{v}\|\,d\Omega\leqslant$$

$$\leqslant\left[|\tau(e_i)|+\frac{\tau(e_i^{(n)})-\tau(e_i^{(n-1)})}{e_i^{(n)}-e_i^{(n-1)}}e_i^{(n-1)}\right]_{x_j=\xi_j}\|\mathbf{u}^{(n)}-\mathbf{u}^{(n-1)}\|_{1\Omega}\|\mathbf{v}\|_{1\Omega}$$

where $\xi=(\xi_1,\xi_2,\xi_3)$ is the point determined by the mean value theorem.

We assume that almost everywhere in the region $\Omega$

$$|\tau(e_i'')|+\frac{\tau(e_i'')-\tau(e_i')}{e_i''-e_i'}e_i'\leqslant\eta<1 \quad (20)$$

We then obtain

$$\|\mathbf{u}^{(n+1)}-\mathbf{u}^{(n)}\|_{1\Omega}\leqslant\eta\|\mathbf{u}^{(n)}-\mathbf{u}^{(n-1)}\|_{1\Omega}\leqslant\eta^n\|\mathbf{u}^{(1)}-\mathbf{u}^{(0)}\|_{1\Omega}$$

It follows from this that $\|\mathbf{u}^{(n+p)}-\mathbf{u}^{(n)}\|_\Omega\to 0$ (since $\|\mathbf{u}\|_\Omega\leqslant\|\mathbf{u}\|_{1\Omega}$) as $n\to\infty$ for any $p$; then by virtue of the completeness of the space, the sequence $\mathbf{u}^{(n)}$ converges to a unique solution $\mathbf{u}\in$ H.

The condition (20) is satisfied, for example, for $\tau\geqslant 0$. Physically, the condition $\tau\geqslant 0$ means that in the problem under consideration the intensity of shear strain is almost everywhere in $\Omega$ no smaller than the value $e_{is}$ determined from the equation $\tau(e_{is})=0$.

In particular, the convergence of the method of elastic solutions [2] follows from (20) when $\tau\equiv\omega\geqslant 0$, i.e. $G^*=G$.

However, the convergence of the sequence $\mathbf{u}^{(n)}$ holds under weaker limitations on $\tau$ than (20). We shall first show that the sequence $\mathbf{u}^{(n)}$ determined by (10) is bounded in H provided that $|\tau|\leqslant\eta<1$.

From Eq. (15) and the existence of a solution in H, it follows that the term independent of $\mathbf{u}$ in (15) is a bounded linear functional in H, and, therefore, may be expressed in the form

$$\int X_j v_j\,d\Omega={}^2/_3\,G^*(\mathbf{f},\mathbf{v})_{1\Omega},\qquad \mathbf{f}\in \text{H}$$

It then follows from the recurrence relation (19) for $\mathbf{v}=\mathbf{u}^{(n+1)}$ that

$$\|\mathbf{u}^{(n+1)}\|_{1\Omega}^2=(\mathbf{f},\mathbf{u}^{(n+1)})_{1\Omega}+\int\tau(\|\mathbf{u}^{(n)}\|)(\mathbf{u}^{(n)},\mathbf{u}^{(n+1)})\,d\Omega\leqslant(\|\mathbf{f}\|_{1\Omega}+\eta\|\mathbf{u}^{(n)}\|_{1\Omega})\|\mathbf{u}^{(n+1)}\|_{1\Omega}$$

We then have

$$\|\mathbf{u}^{(n+1)}\|_{\Omega} \leqslant \|\mathbf{u}^{(n+1)}\|_{1\Omega} \leqslant \eta \|\mathbf{u}^{(n)}\|_{1\Omega} + \|\mathbf{f}\|_{1\Omega} \leqslant \eta^n \|\mathbf{u}^{(1)}\|_{1\Omega} + (1 + \eta + \dots \eta^{n-1})\|\mathbf{f}\|_{1\Omega}$$

which proves the boundedness of the sequence $\mathbf{u}^{(n)}$ in H.

We show further that for all $\tau$ the solution of Eq. (15) is unique. To do this we replace $\tau$ in (15) by $\omega$ and $G^*$ by $G$ in accordance with (6), and assume that two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ exist. Then substituting $\mathbf{u}_1$ and $\mathbf{u}_2$ successively into (15) and setting $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, taking account of (14), (5), and the Schwartz inequality, we have

$$K \int (\theta_{u_2} - \theta_{u_1})^2 \, d\Omega + \frac{2}{3} G \int \|\mathbf{u}_2 - \mathbf{u}_1\|^2 \, d\Omega =$$

$$= \frac{2}{3} G \int [\omega(\|\mathbf{u}_2\|)(\mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) - \omega(\|\mathbf{u}_1\|)(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1)] \, d\Omega =$$

$$= \frac{2}{3} G \int \left[\omega(\mathbf{u}_2)\|\mathbf{u}_2 - \mathbf{u}_1\|^2 + \frac{\omega(\|\mathbf{u}_2\|) - \omega(\|\mathbf{u}_1\|)}{\|\mathbf{u}_2\| - \|\mathbf{u}_1\|}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1)(\|\mathbf{u}_2\| - \|\mathbf{u}_1\|)\right] d\Omega \leqslant$$

$$\leqslant \frac{2}{3} G \int \left[\omega(e_{i2}) + \frac{\omega(e_{i2}) - \omega(e_{i1})}{e_{i2} - e_{i1}} e_{i1}\right] \|\mathbf{u}_2 - \mathbf{u}_1\|^2 \, d\Omega \leqslant \frac{2}{3} G\eta \int \|\mathbf{u}_2 - \mathbf{u}_1\|^2 \, d\Omega$$

Since $\eta < 1$, the inequality holds only for $\|\mathbf{u}_2 - \mathbf{u}_1\| = 0$ almost everywhere in $\Omega$. For the first boundary value problem this means that $\mathbf{u}_1 = \mathbf{u}_2$ almost everywhere in $\Omega$. For the second boundary value problem, the solutions in this case coincide almost everywhere up to rigid-body displacements.

The proof of convergence of the sequence $\mathbf{u}^{(n)}$ will be carried out for an incompressible material, under the conditions

$$\left|\frac{\tau(e_{i2})\, e_{i2} \pm \tau(e_{i1})\, e_{i1}}{e_{i2} \pm e_{i1}}\right| \leqslant \eta < 1, \qquad |\tau| \leqslant \eta < 1 \tag{21}$$

which are satisfied for $G^* > {}^1/_2\, G$. We shall seek the solution of the problem in the class of vector functions for which the condition of incompressibility, $\theta_u = 0$, holds. Then, instead of (15) and (19), we have, respectively

$$\int (\mathbf{u}, \mathbf{v}) \, d\Omega = \int \tau(\|\mathbf{u}\|)(\mathbf{u}, \mathbf{v}) \, d\Omega + \frac{3}{2G^*} \int X_j v_j d\Omega \tag{22}$$

$$\int (\mathbf{u}^{(n+1)}, \mathbf{v}) \, d\Omega = \int \tau(\|\mathbf{u}^{(n)}\|)(\mathbf{u}^{(n)}, \mathbf{v}) \, d\Omega + \frac{3}{2G^*} \int X_j v_j \, d\Omega \tag{23}$$

The following theorem on the convergence of the successive approximation process holds.

T h e o r e m . Under the assumptions made above regarding $X_j$ and the choice of the initial element, the sequence (23) converges in the space H to the solution of Eq. (22) provided that the condition

$$G^* > {}^1/_2\, G \tag{24}$$

is satisfied.

P r o o f. We introduce a representation of the functions $(\mathbf{u}, \mathbf{v})$ with the aid of their means. Let $\mathbf{u}$ be extended beyond $\Omega$ by assigning it the value zero there and let $h(x, \xi)$ be an averaging kernel in the circle $C$ of radius $\rho$ with center at the point $x \in \Omega$, the kernel being continuously differentiable in $C$ the required number of times and equal to zero outside $C$. Then we have

$$(\mathbf{u}, \mathbf{v}) = \lim_{\rho \to 0} (\mathbf{u}, \mathbf{v})_\rho, \qquad (\mathbf{u}, \mathbf{v})_\rho = \int (\mathbf{u}(\xi), \mathbf{v}(\xi))\, h(x, \xi) \, dC, \qquad \xi \in C$$

Here the limit is to be undestood in the sense of the metric of the space of summable functions.

Since by assumption $\int X_j v_j \, d\Omega$ is a bounded functional in H, it can be represented in the form $(\mathbf{g}, \mathbf{v})_\Omega$. Then (22) can be represented in the form

$$\int \lim_{\rho\to 0} (\mathbf{u}, \mathbf{v})_\rho \, d\Omega = \int \tau(\|\mathbf{u}\|) \lim_{\rho\to 0} (\mathbf{u}, \mathbf{v})_\rho \, d\Omega + \frac{3}{2G^*} \int \lim_{\rho\to 0} (\mathbf{g}, \mathbf{v})_\rho \, d\Omega \tag{25}$$

A much more general meaning can be attributed to the relation that has been obtained than in the case of (22) if the vector functions here are considered as functions of the two variables $\xi$ and $x$. To this end, we introduce a scalar product according to the formula

$$(\mathbf{u}, \mathbf{v})_{2\Omega} = \int (\mathbf{u}, \mathbf{v})_2 \, d\Omega, \quad (\mathbf{u}, \mathbf{v})_2 = \lim_{\rho\to 0} (\mathbf{u}(x, \xi), \mathbf{v}(x, \xi))_\rho, \quad \xi \in C, \ x \in \Omega$$

(the product $(\mathbf{u}, \mathbf{v})_2$ coincides with $(\mathbf{u}, \mathbf{v})$ if $\mathbf{u}$ and $\mathbf{v}$ are functions of $\xi$ alone) and consider the corresponding Hilbert space $H_2$ of vector functions $\mathbf{u}$ defined in $\Omega \times \Omega$ for which a finite norm $\|\mathbf{u}\|_{2\Omega}$ exists. By definition, the space H is imbedded in $H_2$. It is easy to prove that in $H_2$ there exists a solution of the equation

$$\int (\mathbf{u}, \mathbf{v})_2 \, d\Omega = \int \tau(\|\mathbf{u}\|_2) (\mathbf{u}, \mathbf{v})_2 \, d\Omega + \frac{3}{2G^*} \int (\mathbf{g}, \mathbf{v})_2 \, d\Omega \tag{26}$$

if by $\tau(\|\mathbf{u}\|_2)$ is meant a function with properties analogous to those of $\tau(\|\mathbf{u}\|)$ and $\mathbf{g} = \mathbf{g}(x, \xi)$. For this it is sufficient to replace $\tau$ and $G^*$ by $\omega$ and $G$, respectively, in (26) according to (6) and to form a sequence of the form

$$\int (\mathbf{u}^{(n+1)}, \mathbf{v})_2 \, d\Omega = \int \omega(\|\mathbf{u}^{(n)}\|_2) (\mathbf{u}^{(n)}, \mathbf{v})_2 \, d\Omega + \frac{3}{2G} \int (\mathbf{g}, \mathbf{v})_2 \, d\Omega$$

If the initial element of the sequence $\mathbf{u}^{(0)}$ and $\mathbf{g}$ are selected so that $(\mathbf{u}^{(0)}, \mathbf{v})_{2\Omega}$ and $(\mathbf{g}, \mathbf{v})_{2\Omega}$ are bounded functionals in $H_2$, then all the $(\mathbf{u}^{(n)}, \mathbf{v})_{2\Omega}$ will be bounded functionals in $H_2$ and $\mathbf{u}^{(n)}$ exists in $H_2$. The convergence of the sequence to the unique solution of Eq. (26) is proved in exactly the same way as for the case of the ordinary method of elastic solutions. Equation (26) defines a certain operator $T$: $g(x, \xi) \to \mathbf{u}(x, \xi)$. If $\mathbf{g}(x, \xi) = \mathbf{g}(\xi)$ and $\mathbf{g}(\xi)$ is chosen from Eq.(25) then the operator $T$ determines $\mathbf{u}^*(\xi)$, which is the solution of Eq. (25). For by setting $\mathbf{v} = \mathbf{v}(\xi)$ in (26) we obtain Eq. (25) and because of the uniqueness of solutions of Eqs. (25) and (26), the solution of Eq. (26) will coincide in this case with the solution of Eq. (25). Let us now show that the sequence of the form

$$\int (\mathbf{u}^{(n+1)}, \mathbf{v})_2 \, d\Omega = \int \tau(\|\mathbf{u}^{(n)}\|_2) (\mathbf{u}^{(n)}, \mathbf{v})_2 \, d\Omega + \frac{3}{2G^*} \int (\mathbf{g}, \mathbf{v})_2 \, d\Omega \tag{27}$$

$$\mathbf{g} = \mathbf{g}(\xi), \quad \mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\xi)$$

converges to the solution of Eq. (26).

The existence of the sequence $\mathbf{u}^{(n)}$ follows from the boundedness of the functionals $(\mathbf{u}^{(0)}, \mathbf{v})_{2\Omega}$ and $(\mathbf{g}, \mathbf{v})_{2\Omega}$ in $H_2$ and consequently of all the $(\mathbf{u}^{(n)}, \mathbf{v})_{2\Omega}$. By uniqueness, this sequence coincides with the sequence (23).

Let us represent a vector function in H in the form

$$\mathbf{u} = \boldsymbol{\varphi} + \boldsymbol{\psi} \tag{28}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are defined as follows. At a point in the neighborhood of which $(\mathbf{u}^*, \mathbf{u}^*) \neq 0$, almost everywhere, we set

$$\boldsymbol{\varphi}(x, \xi) = \frac{(\mathbf{u}, \mathbf{u}^*)}{(\mathbf{u}^*, \mathbf{u}^*)} \mathbf{u}^*(\xi), \qquad \boldsymbol{\psi} \equiv \mathbf{u} - \boldsymbol{\varphi}$$

and for points where $(\mathbf{u}^*, \mathbf{u}^*) = 0$, we set $\boldsymbol{\varphi} = 0$, $\boldsymbol{\psi} = \mathbf{u}$. It is then easy to verify that the representation (28) has the following properties

$$(\mathbf{u}^*, \boldsymbol{\psi})_2 = 0, \quad (\boldsymbol{\varphi}, \boldsymbol{\psi})_2 = 0, \quad (\mathbf{u}^*, \boldsymbol{\varphi})_2 = \pm \|\mathbf{u}^*\|_2 \|\boldsymbol{\varphi}\|_2, \quad \|\mathbf{u}\|_2^2 = \|\boldsymbol{\varphi}\|_2^2 + \|\boldsymbol{\psi}\|_2^2$$

It is clear from the last one that $\|\boldsymbol{\varphi}\|_2^2$ and $\|\boldsymbol{\psi}\|_2^2$ are summable functions.

In (26) and (27) we set $\mathbf{v} = \boldsymbol{\varphi}^{(n+1)}$. Taking account of the properties of the decomposition we have

$$\int (\mathbf{g}, \psi^{(n+1)})_2\, d\Omega = 0, \qquad \| \psi^{(n+1)} \|_{2\Omega}^2 = \int \tau (\| \mathbf{u}^{(n)} \|_2) (\psi^{(n)}, \psi^{(n+1)})_2\, d\Omega \leqslant$$

$$\leqslant \eta \int | (\psi^{(n+1)}, \psi^{(n)})_2 |\, d\Omega \leqslant \eta \| \psi^{(n+1)} \|_{2\Omega} \| \psi^{(n)} \|_{2\Omega}.$$

From this

$$\| \psi^{(n+1)} \|_{2\Omega} \leqslant \eta \| \psi^{(n)} \|_{2\Omega} \leqslant \eta^n \| \psi^{(1)} \|_{2\Omega} \to 0 \quad n \to \infty$$

that is, as $n \to \infty$, the sequence $\mathbf{u}^{(n)}$ tends to become "parallel" to $\mathbf{u}^*$.

In accordance with (26) and (27) we have for the sequence $\mathbf{u}^{(n)}$

$$\int (\mathbf{u}^{(n+1)} - \mathbf{u}^*, \mathbf{v})_2\, d\Omega = \int [\tau (\| \mathbf{u}^{(n)} \|_2) (\mathbf{u}^{(n)}, \mathbf{v})_2 - \tau (\| \mathbf{u}^* \|_2) (\mathbf{u}^*, \mathbf{v})_2]\, d\Omega$$

Setting $\mathbf{v} = \mathbf{u}^{(n+1)} - \mathbf{u}^*$, using the representation (28), and taking account of (8) we have

$$\int [\tau (\| \mathbf{u}^{(n)} \|_2) (\mathbf{u}^{(n)}, \mathbf{u}^{(n+1)} - \mathbf{u}^*)_2 - \tau (\| \mathbf{u}^* \|_2) (\mathbf{u}^*, \mathbf{u}^{(n+1)} - \mathbf{u}^*)_2]\, d\Omega =$$

$$= \int [\tau (\| \mathbf{u}^{(n)} \|_2) - \tau (\| \varphi^{(n)} \|_2)] (\varphi^{(n)}, \varphi^{(n+1)} - \mathbf{u}^*)_2\, d\Omega + \int \tau (\| \mathbf{u}^{(n)} \|_2) (\psi^{(n)}, \psi^{(n+1)})_2\, d\Omega +$$

$$+ \int \frac{\tau (\| \varphi^{(n)} \|_2) (\varphi^{(n)}, \varphi^{(n+1)} - \mathbf{u}^*)_2 - \tau (\| \mathbf{u}^* \|_2) (\mathbf{u}^*, \varphi^{(n+1)} - \mathbf{u}^*)_2}{(\varphi^{(n)}, \varphi^{(n+1)} - \mathbf{u}^*)_2 - (\mathbf{u}^*, \varphi^{(n+1)} - \mathbf{u}^*)_2} (\varphi^{(n)} - \mathbf{u}^*, \varphi^{(n+1)} - \mathbf{u}^*)_2\, d\Omega =$$

$$= \pm \int \frac{\tau (\| \mathbf{u}^{(n)} \|_2) - \tau (\| \varphi^{(n)} \|_2)}{\| \mathbf{u}^{(n)} \|_2 - \| \varphi^{(n)} \|_2} (\sqrt{\| \varphi^{(n)} \|_2^2 + \| \psi^{(n)} \|_2^2} - \| \varphi^{(n)} \|_2^2) \| \varphi^{(n)} \|_2 \| \varphi^{(n+1)} - \mathbf{u}^* \|_2\, d\Omega \pm$$

$$\pm \int \frac{\tau (\| \varphi^{(n)} \|_2 (\pm \| \varphi^{(n)} \|_2) - \tau (\| \mathbf{u}^* \|_2) (\pm \| \mathbf{u}^* \|_2)}{(\pm \| \varphi^{(n)} \|_2) - (\pm \| \mathbf{u}^* \|_2)} \| \varphi^{(n)} - \mathbf{u}^* \|_2 \| \varphi^{(n+1)} - \mathbf{u}^* \|_2\, d\Omega +$$

$$+ \int \tau (\| \mathbf{u}^{(n)} \|_2) (\psi^{(n)}, \psi^{(n+1)})_2\, d\Omega \leqslant 2\eta \int (\sqrt{\| \varphi^{(n)} \|_2^2 + \| \psi^{(n)} \|_2^2} - \| \varphi^{(n)} \|_2) \| \varphi^{(n+1)} - \mathbf{u}^* \|_2\, d\Omega +$$

$$+ \eta \int \| \varphi^{(n)} - \mathbf{u}^* \|_2 \| \varphi^{(n+1)} - \mathbf{u}^* \|_2\, d\Omega + \eta \int \| \psi^{(n)} \|_2 \| \psi^{(n+1)} \|_2\, d\Omega \leqslant$$

$$\leqslant 2c_1\eta \| \varphi^{(n)} - \mathbf{u}^* \|_{2\Omega} \| \psi^{(n)} \|_{2\Omega} + \eta \| \psi^{(n)} \|_{2\Omega} \| \psi^{(n+1)} \|_{2\Omega} + \eta \| \varphi^{(n)} - \mathbf{u}^* \|_{2\Omega} \| \varphi^{(n+1)} - \mathbf{u}^* \|_{2\Omega}$$

since

$$\int (\sqrt{\| \varphi^{(n)} \|_2^2 + \| \psi^{(n)} \|_2^2} - \| \psi^{(n)} \|_2)^2\, d\Omega \leqslant c_1^2 \int \| \psi^{(n)} \|_2^2\, d\Omega, \qquad \left| \frac{\tau_2 - \tau_1}{e_{i2} - e_{i1}} e_{i1} \right| < 2\eta$$

Thus we have

$$\| \varphi^{(n+1)} - \mathbf{u}^* \|_{2\Omega}^2 + \| \psi^{(n+1)} \|_{2\Omega}^2 \leqslant \eta \| \varphi^{(n)} - \mathbf{u}^* \|_{2\Omega} \| \varphi^{(n+1)} - \mathbf{u}^* \|_{2\Omega} + c_2 \| \psi^{(n)} \|_{2\Omega}$$

It follows from this that for sufficiently large $n$ either

$$\| \varphi^{(n+1)} - \mathbf{u}^* \|_{2\Omega} < \varepsilon \qquad (\varepsilon \text{ is a sufficiently small quantity})$$

or

$$\| \varphi^{(n+1)} - \mathbf{u}^* \|_{2\Omega} \leqslant \eta_1 \| \varphi^{(n)} - \mathbf{u}^* \|_{2\Omega}, \qquad \eta_1 < 1$$

Assuming that the first relation, the desired one, is never satisfied, we arrive at the result

$$\| \varphi^{(n+p+1)} - \mathbf{u}^* \|_{2\Omega} \leqslant \eta_1^p \| \varphi^{(n)} - \mathbf{u}^* \|_{2\Omega}$$

and as $p \to \infty$ we have

$$\| \varphi^{(n+p+1)} - \mathbf{u}^* \|_{2\Omega} \to 0$$

The theorem is then proved.

It is easy to see from (27) that the process diverges if $-\tau \geqslant \eta > 1$. For setting $\mathbf{v} = \psi^{(n)}$, we obtain

$$\eta \| \psi^{(n)} \|_{2\Omega}^2 \leqslant \left| \int \tau (\| \mathbf{u}^{(n)} \|_2) \| \psi^{(n)} \|_2^2\, d\Omega \right| = \left| \int (\psi^{(n+1)}, \psi^{(n)})_2\, d\Omega \right| \leqslant \| \psi^{(n+1)} \|_{2\Omega} \| \psi^{(n)} \|_{2\Omega}$$

and then

$$\| \psi^{(n+1)} \|_{2\Omega} \geqslant \eta \| \psi^{(n)} \|_{2\Omega} \geqslant \eta^n \| \psi^{(1)} \|_{2\Omega} \to \infty, \qquad n \to \infty$$

At the same time it is clear from the proof of the theorem that the condition $\eta < 1$ in (8) need not be satisfied everywhere in $\Omega$. In particular, this condition can be violated on a manifold of smaller dimensionality than that to which the solution belongs.

The convergence of the process with respect to the norm of the space H occurs more rapidly the smaller $\eta$ is. It follows from (8) that $\eta$ is smallest for $| 1 - G / G^* | = | 1 - a / 3G^* | < 1$, where $a$ is the smallest value of the slope of the $\sigma_i$ vs. $e_i$ curve in the interval of convergence of the process.

We remark that the proof of convergence is also valid in the case when effective moduli are chosen for each step of the process. In this case they should satisfy the condition $1/2\, G < A \leqslant G_n \leqslant B < \infty$.

We note further that for negative $\tau$, which correspond to small $G^*$, a nonmonotonic sequence of approximations in the space $H_2$ is obtained. This can easily be seen from (27) if v is set equal to $\psi^{(n)}$, since then the scalar product $(\psi^{(n)}, \psi^{(n+1)})_{2\Omega}$ is negative. This circumstance permits us to make a "two-sided" estimate of the solution with the aid of two successive approximations.

The proof which has been presented does not cover the case in which the scalar product (13) degenerates into an ordinary product. However, in this case the method of proof is made apparent by the example given below, in which consideration of the compressibility makes it possible to obtain a less restrictive condition of convergence than (24).

As an example let us examine the convergence of the process in the case of symmetrical deformation of a sphere. In the present case, the equation which is analogous to (9) has the form

$$\left(K + \frac{4}{3} G^*\right) \frac{d}{dr} \frac{1}{r^2} \frac{dr^2 u}{dr} = \frac{4}{3} G^* \frac{1}{r^3} \frac{d}{dr} \tau r^4 \frac{d}{dr} \frac{u}{r} - R(r), \quad u|_\Gamma = 0 \tag{29}$$

where $u(r)$ is the radial displacement and $R(r)$ is the radial body force. The sequence of approximate solutions is defined in the form

$$\left(K + \frac{4}{3} G^*\right) \frac{d}{dr} \frac{1}{r^2} \frac{dr^2 u_{n+1}}{dr} = \frac{4}{3} G^* \frac{1}{r^3} \frac{d}{dr} \tau_n r^4 \frac{d}{dr} \frac{u_n}{r} - R(r), \quad u_{n+1}|_\Gamma = 0 \tag{30}$$

Multiplying (30) by a continuously differentiable function $v$, $v|_\Gamma = 0$, and integrating over the interval $[a, b]$, the region of definition of the solution, we obtain, taking account of (6)

$$\left(K + \frac{4}{3} G^*\right) \int_a^b \frac{1}{r} \frac{dr^2 (u_{n+1} - u_n)}{dr} \frac{1}{r} \frac{dr^2 v}{dr} dr =$$

$$= \frac{4}{3} G^* \int_a^b \left(\tau_n r^2 \frac{d}{dr} \frac{u_n}{r} - \tau_{n-1} r^2 \frac{d}{dr} \frac{u_{n-1}}{r}\right) r^2 \frac{d}{dr} \frac{v}{r} dr = \tag{31}$$

$$= \frac{4}{3} G^* \int_a^b \frac{\tau_n e_{in} \pm \tau_{n-1} e_{in-1}}{e_{in} \pm e_{in-1}} r^2 \frac{d}{dr} \frac{u_n - u_{n-1}}{r} r^2 \frac{dv}{dr} dr, \quad e_i = \frac{2}{3} \left| r \frac{d}{dr} \frac{u}{r} \right|$$

Setting $v = u_{n+1} - u_n$ we obtain according to (21)

$$\left(K + \frac{4}{3} G^*\right) \int_a^b \left[\frac{1}{r} \frac{dr^2 (u_{n+1} - u_n)}{dr}\right]^2 dr = \left(K + \frac{4}{3} G^*\right) \int_a^b \left(r^2 \frac{d}{dr} \frac{u_{n+1} - u_n}{r}\right)^2 dr \leqslant$$

$$\leqslant \frac{4}{3} G^*\eta \left[\int_a^b \left(r^2 \frac{d}{dr} \frac{u_n - u_{n-1}}{r}\right)^2 dr \int_a^b \left(r^2 \frac{d}{dr} \frac{u_{n+1} - u_n}{r}\right)^2 dr\right]^{1/2}$$

Then, introducing the definition

$$\| u \|_\Omega^2 = \int_a^b \left(r \frac{d}{dr} \frac{u}{r}\right)^2 dr$$

we obtain

$$\| u_{n+1} - u_n \|_\Omega \leqslant \frac{4G^*\eta}{3K + 4G^*} \| u_n - u_{n-1} \|_\Omega = \lambda \| u_n - u_{n-1} \|_\Omega \qquad \lambda = \frac{4G^*\eta}{3K + 4G^*}$$

i.e. the process converges like a geometric progression if $\lambda < 1$. The condition $\lambda < 1$ can be satisfied even for $\eta \geqslant 1$, for example, in cases where there is a flow region in the $\sigma_i$ $(e_i)$ curve. On the other hand, by choosing $\eta$ equal to the left-hand side of the inequality (8), we find that the following inequalities are consequences of $\lambda < 1$ :

$$\frac{4(G - G^*)}{3K + 4G^*} < 1, \qquad G^* > \frac{1 - 5\nu}{1 - 2\nu} \frac{G}{4} \tag{32}$$

where $\nu$ is Poisson's ratio. It is clear from this that in the present case convergence occurs even for $G^* < 1/_2\, G$.

The condition (32) for $G^*$ is necessary. For let us assume that $u$ is the elastic solution. Then $\omega = 0$ and, setting $u_{n-1} = u$ in (31), we have from (7) that

$$\left(K + \frac{4}{3} G^*\right) \int_a^b \left(r^2 \frac{d}{dr} \frac{u_{n+1} - u}{r}\right)^2 dr =$$

$$= -\frac{4}{3} \int_b^b (G - G^* - G\omega_n) r^2 \frac{d}{dr} \frac{u_n - u}{r} r^2 \frac{d}{dr} \frac{u_{n+1} - u}{r} dr$$

It is clear from this that the negative $\tau$ the approximation process is nonmonotonic. The successive approximations are then obtained as upper and lower estimates of the solution. Moreover the sequence will diverge if

$$\frac{4(G - G^*)}{3K + 4G^*} \geqslant \lambda > 1$$

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## BIBLIOGRAPHY

1. Il'iushin, A. A., Plasticity. Moscow-Leningrad, Gostekhizdat, 1948.
2. Vorovich, I. I. and Krasovskii, Iu. P., On the method of elastic solutions. Dokl. Akad. Nauk SSSR, Vol. 126, №4, 1959.
3. Krasovskii, Iu. P., Solvability of the two-dimensional problem of small elastic-plastic strains. Dokl. Akad. Nauk SSSR, Vol. 126, №5, 1959.
4. Mikhlin, S. G., Direct Methods in Mathematical Physics, Moscow-Leningrad, Gostekhizdat, 1950.
5. Sobolev, S. L., Some Applications of Functional Analysis in Mathematical Physics. Izd. Leningr. Univ. (Leningr. Univ. Press), 1950.

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